Mathematical analysis of a class of path-dependent options

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ISCAMI, Bratislava 2010

Obsah

Financial derivatives

- Financial derivatives are financial instruments that are linked to a specific financial instrument or indicator or commodity, and which provide for market financial risk in a form that can be traded or otherwise offset in the market.
- The value of the financial derivative derives from the price of the underlying items.

Usage

Financial derivatives are used for a number of purposes including risk management, hedging and speculation. Between financial derivatives belongs e.q. futures, forwards, **options** and swaps.

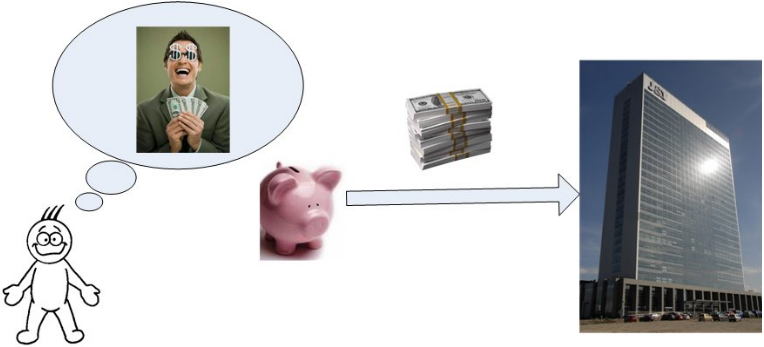
Options

- Option is a right (**but not obligation**) to buy or sell an asset by a certain date for a predetermined price X.
- can be exercised before expiration date?
  - yes - American-style option
  - no - European-style option

Options' types

- plain-vanilla call $V = (S - X)^+$
- floating strike Assian call $V = (S - A)^+$
- floating rate Assian call $V = (A - X)^+$
- lookback options $V = (S - m)^+$

How does it work?



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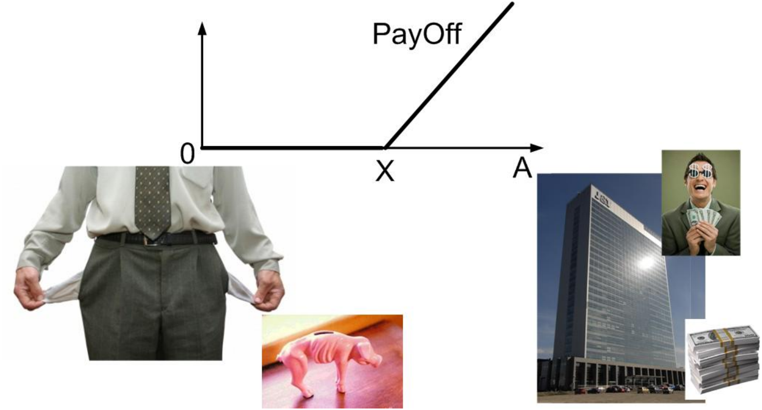
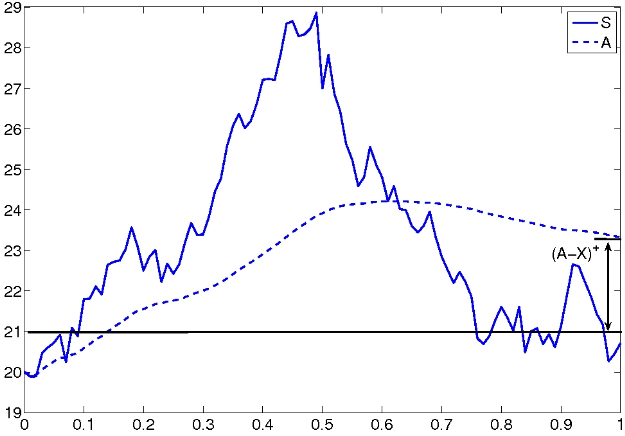
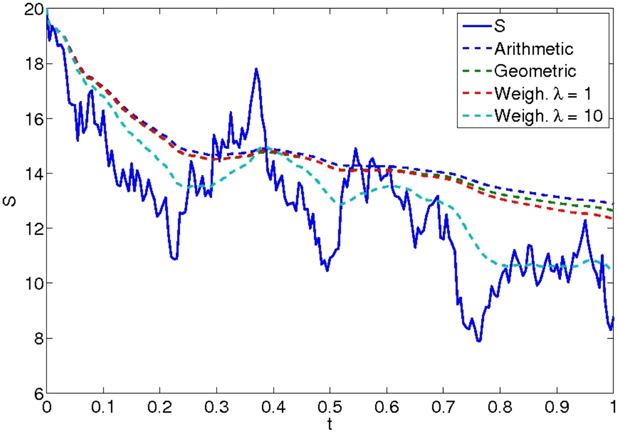




A holder of the Asian option can be secured for the risk arising from a sudden underlying price jumps



Geometric average	Arithmetic average	Weighted arithmetic average
$\ln A_T = \frac{1}{T}\int_0^T \ln S_\xi \; d\xi$	$A_T = \frac{1}{T}\int_0^T S_\xi \; d\xi$	$A_T = \frac{1}{\int_0^T a(\xi) d\xi}\int_0^T a(T-\xi) S_\xi \; d\xi$ $a(\xi) = \exp(-\lambda\xi)$

PDE for Asian options

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} + Af\left(\frac{S}{A}, t\right) \frac{\partial V}{\partial A} - rV = 0,$$

$$V(S, A, t) = \max(S - A, 0),$$

where $S, A > 0,\ t \in (0, T)$ and

$$f(x,t) = \begin{cases} \dfrac{x-1}{t}, & \text{for arithmetic avg.,} \\ \dfrac{\lambda(x-1)}{1 - exp(-\lambda t)}, & \text{for exp. weig. avg.,} \\ \dfrac{\ln(x)}{t}, & \text{for geometric avg..} \end{cases} \tag{1}$$

Dimension reduction

- new state variable $x = \dfrac{S}{A}$
- new function $W(x,\tau) = \dfrac{1}{A} V(S, A, t)$, where $\tau = T - t$.

$$
\begin{aligned}
0 &= \frac{\partial W}{\partial \tau} + [f(x, T-\tau) - r + q]\, x \frac{\partial W}{\partial x} \\
&\quad - \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (r - f(x, T-\tau)) W, \\
W(x,0) &= \max(x-1, 0),
\end{aligned}
$$

where $\tau \in (0, T),\ x > 0$.

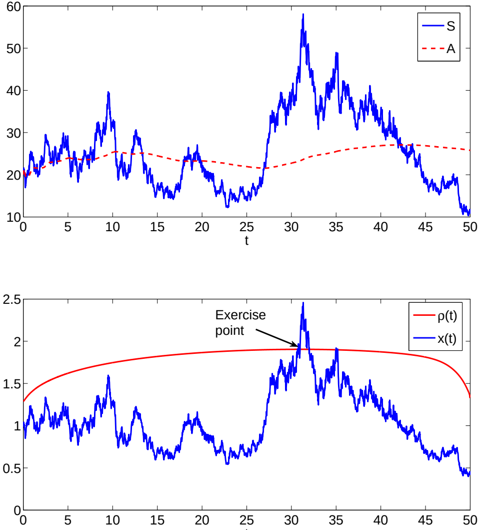
American style - Asian options with early exercise possibility

Free boundary problem

American style

- $0 < x < \rho(\tau), \tau \in (0, T)$,
- $W(0, \tau) = 0$,
- $W(x, \tau) = x - 1$, (terminal condition)
- $\frac{\partial W}{\partial x}(x, \tau) = 1$, at $x = \rho(\tau)$, (smooth pasting principle)
- **spatial domain depends on boundary function ρ.**

Early exercise boundary



Fixed domain transformation

- $\xi = \ln\left(\frac{\rho(\tau)}{x}\right)$
- $x \in (0, \rho(\tau))$ iff $\xi \in (0, \infty)$ for $\tau \in (0, T)$
- $\Pi(\xi, \tau) = W(x, \tau) - x\frac{\partial W}{\partial x}(x, \tau)$ (financial meaning – synthetised portfolio)

$$
\begin{aligned}
0 &= \frac{\partial \Pi}{\partial \tau} + \left[\frac{\dot{\rho}}{\rho} - f(\rho e^{-\xi}, T - \tau) + r - q - \frac{\sigma^2}{2}\right]\frac{\partial \Pi}{\partial \xi} - \frac{\sigma^2}{2}\frac{\partial^2 \Pi}{\partial \xi^2} \\
&\quad + \left[r + x\frac{\partial f}{\partial x} - f(x, T - \tau)\Big|_{x = \rho e^{-\xi}}\right]\Pi.
\end{aligned}
$$

$$
\Pi(\xi, 0) = \begin{cases} -1, & \xi < \ln\rho(0), \\ 0, & \xi > \ln\rho(0). \end{cases}, \; \Pi(0, \tau) = -1, \; \Pi(\infty, \tau) = 0.
$$

Algebraic constraint between $\rho(\tau)$ and $\Pi(\xi,\tau)$

Derivative form

$$q\rho(\tau) - r - \frac{\sigma^2}{2}\frac{\partial \Pi}{\partial \xi}(0,\tau) + f(\rho(\tau), T-\tau) = 0.$$

Integral form

$$0 = \frac{d}{d\tau}\left(\ln \rho(\tau) + \int_0^\infty \Pi(\xi,\tau)d\xi\right) + q\rho(\tau) - q - \frac{\sigma^2}{2} + \int_0^\infty (r - f(\rho(\tau)e^{-\xi}, T-\tau))\Pi(\xi,\tau)d\xi.$$

Algebraic constraint between $\rho(\tau)$ and $\Pi(\xi,\tau)$

Derivative form

$$q\rho(\tau) - r - \frac{\sigma^2}{2}\frac{\partial\Pi}{\partial\xi}(0,\tau) + f(\rho(\tau), T-\tau) = 0.$$

Integral form

$$\begin{aligned} 0 \;=\; & \frac{d}{d\tau}\left(\ln\rho(\tau) + \int_0^\infty \Pi(\xi,\tau)d\xi\right) + q\rho(\tau) - q - \frac{\sigma^2}{2} \\ & + \int_0^\infty (r - f(\rho(\tau)e^{-\xi}, T-\tau))\Pi(\xi,\tau)d\xi. \end{aligned}$$

Characterization

- discretization
- restrict spatial domain to a finite interval $\xi \in (0, L)$
- $k = \frac{T}{m}$ - discretization time step
- $h = \frac{L}{n} > 0$ - spatial step
- $\tau_j = jk$
- Π_i^j - full space-time approximation of $\Pi(\xi_i, \tau_j)$
- $\rho^j = \rho(\tau_j)$

Euler backward in time finite difference approximation

$$
\begin{aligned}
0 \quad = \quad & \frac{\Pi^j - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^j}{\partial \xi} - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi}, T - \tau) \right) \frac{\partial \Pi^j}{\partial \xi} \\
& - \frac{\sigma^2}{2} \frac{\partial^2 \Pi}{\partial \xi^2} + \left[r + x \frac{\partial f}{\partial x} - f(x, T - \tau) \Big|_{x = \rho^j e^{-\xi}} \right] \Pi^j,
\end{aligned}
$$

where $c(\tau) = \frac{\dot{\rho}(\tau)}{\rho(\tau)} + r - q$.

Operator splitting method

- auxiliary intermediate step $\Pi^{j-\frac{1}{2}}$
- Convection part

$$\frac{\Pi^{j-\frac{1}{2}} - \Pi^{j-1}}{k} + c^j \frac{\partial \Pi^{j-\frac{1}{2}}}{\partial \xi} = 0$$

- Diffusive part

$$\begin{aligned} 0 &= \frac{\Pi^j - \Pi^{j-\frac{1}{2}}}{k} - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi}, T-\tau) \right) \frac{\partial \Pi^j}{\partial \xi} - \frac{\sigma^2}{2} \frac{\partial^2 \Pi^j}{\partial \xi^2} \\ &+ \left[r + x \frac{\partial f}{\partial x} - f(x, T-\tau) \Big|_{x=\rho^j e^{-\xi}} \right] \Pi^j \end{aligned}$$

Transport equation

Approximation of convective part

$$\frac{\partial \tilde{\Pi}}{\partial \tau} + c(\tau)\frac{\partial \tilde{\Pi}}{\partial \xi} = 0,$$

for $\xi > 0$ and $\tau \in (\tau_{j-1}, \tau_j\rangle$

- initial condition $\tilde{\Pi}(\xi, \tau_{j-1}) = \Pi^{j-1}(\xi)$
- boundary condition $\tilde{\Pi}(0, \tau) = -1$.

Operator $\mathcal{T}$

$$\Pi_i^{j-\frac{1}{2}} = \begin{cases} \Pi^{j-1}(\nu_i), & \text{if } \nu_i = \xi_i + \ln \frac{\rho^{j-1}}{\rho^j} - (r-q)k > 0, \\ -1, & \text{otherwise.} \end{cases}$$

A comparison of different interpolation methods in operator $\mathcal{T}$

m	interp. method	ε_ρ	ε_{Π_1}	ε_{Π_2}
200	linear	0.355520	0.008903	0.379569
	spline	0.355506	0.008234	0.378894
	cubic	0.167454	0.008234	0.378894
400	linear	0.151797	0.007341	0.190087
	spline	0.151780	0.007878	0.188898
	cubic	0.137178	0.007897	0.188898
800	linear	0.052020	0.006184	0.087705
	spline	0.052014	0.007696	0.083035
	cubic	0.049852	0.007702	0.083044
1600	linear	0.027937	0.004279	0.034823
	spline	0.027936	0.005383	0.027244
	cubic	0.010368	0.005394	0.027352

linear - Linear interpolation; **spline** - Cubic spline interpolation; **cubic** - Piecewise cubic Hermite interpolation.

$\varepsilon_\rho = ||\rho - \rho_{benchmark}||_\infty$, $\varepsilon_{\Pi_1} = ||\Pi(\cdot, 25) - \Pi_{benchmark}(\cdot, 25)||_\infty$,

$\varepsilon_{\Pi_2} = ||\Pi(\cdot, 49.5) - \Pi_{benchmark}(\cdot, 49.5)||_\infty$.

Finite differences approximation of diffusive part

$$\begin{aligned} 0 &= \frac{\Pi_i^j - \Pi_i^{j-\frac{1}{2}}}{k} + \left[r + x\frac{\partial f}{\partial x} - f(x, T-\tau)\Big|_{x=\rho^j e^{-\xi_i}} \right] \Pi_i^j \\ &\quad - \left(\frac{\sigma^2}{2} + f(\rho^j e^{-\xi_i}, T-\tau) \right) \frac{\Pi_{i+1}^j - \Pi_{i-1}^j}{2h} - \frac{\sigma^2}{2} \frac{\Pi_{i+1}^j - 2\Pi_i^j + \Pi_{i-1}^j}{h^2}. \end{aligned}$$

Operator A

Tridiagonal system of linear equations

$$\alpha_i^j \Pi_{i-1}^j + \beta_i^j \Pi_i^j + \gamma_i^j \Pi_{i+1}^j = \Pi_i^{j-\frac{1}{2}},$$

$$\begin{aligned} \alpha_i^j(\rho^j) &= -\frac{k}{2h^2}\sigma^2 + \frac{k}{2h}\left(\frac{\sigma^2}{2} + f(\rho^j e^- \xi_i, T-\tau_j) \right), \\ \beta_i^j(\rho^j) &= 1 + b(\xi_i, T-\tau_j)k - (\alpha_i^j + \gamma_i^j), \\ \gamma_i^j(\rho^j) &= -\frac{k}{2h^2}\sigma^2 - \frac{k}{2h}\left(\frac{\sigma^2}{2} + f(\rho^j e^- \xi_i, T-\tau_j) \right), \end{aligned}$$

Algebraic constraint

$$\begin{aligned}\ln \rho^{j} &= \ln \rho^{j-1} + \int_0^\infty \Pi^{j-1}(\xi)d\xi - \int_0^\infty \Pi^{j}(\xi)d\xi \\ &+k\left(q + \frac{\sigma^2}{2} - q\rho^{j-1} - \int_0^\infty \left(r - f(\rho^{j-1}e^{-\xi}, T-\tau_j)\right)\Pi^j d\xi\right). \end{aligned} \tag{2}$$

Operator $\mathcal{F}$

$\ln \mathcal{F}(\Pi^j)$ is right side of equation (2).

Successive iterations procedure

- for $j \geq 1$: $\Pi^{j,0} = \Pi^{j-1}$, $\rho^{j,0} = \rho^{j-1}$.
- $(p+1)$-th approximation of Π^j and ρ^j:

$$\begin{aligned}
\rho^{j,p+1} &= \mathcal{F}(\Pi^{j,p}), && (3)\\
\Pi^{j-\frac{1}{2},p+1} &= \mathcal{T}(\rho^{j,p+1}), && (4)\\
\mathcal{A}(\rho^{j}_{p+1})\Pi^{j,p+1} &= \Pi^{j-\frac{1}{2},p+1}. && (5)
\end{aligned}$$

Algorithm

Input variables: $q \geq 0, L, r, \sigma, n, m, T > 0, \lambda > 0,\ p_{max} = 500, toll = 10^{-8}$
Initialization:

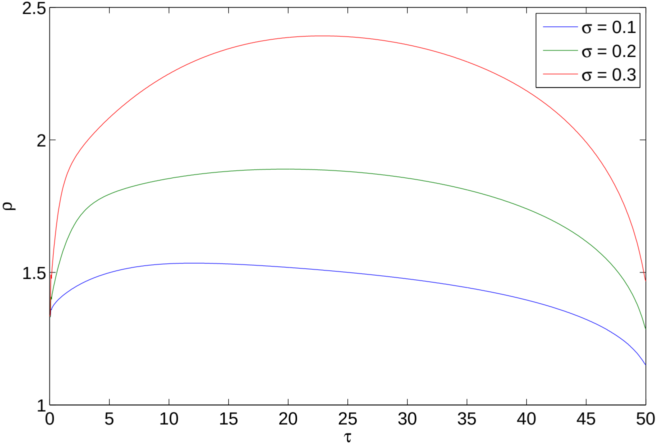
$$k = T/m$$
$$h = L/n$$
$$\rho^0 = \begin{cases} \max\{\frac{1+rT}{1+qT}, 1\}, & \text{for arithmetic average,} \\ \max\{\tilde{x}, 1\}, & \text{for geometric average,} \\ \max\{\frac{\lambda+r(1-e^{-\lambda T})}{\lambda+q(1-e^{-\lambda T})}, 1\}, & \text{for exponentially weighted average,} \end{cases}$$
$$\Pi^0 = \begin{cases} -1, & \xi < \ln \rho^0, \\ 0, & \xi > \ln \rho^0, \end{cases}$$

for $j = 1$ **to** m:

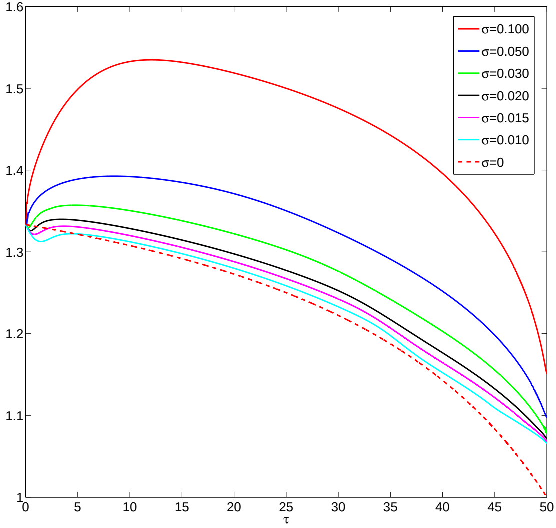
$\Pi^j_0 = \Pi^{j-1}$
$\rho^j_0 = \rho^{j-1}$
for $p = 0$ **to** p_{max}:
$\rho^j_{p+1} = \mathcal{F}(\Pi^j_p)$
$\Pi^{j-\frac{1}{2}}_{p+1} = \mathcal{T}(\rho^j_{p+1})$
$\mathcal{A}(\rho^j_{p+1})\Pi^j_{p+1} = \Pi^{j-\frac{1}{2}}_{p+1}$
if $(|\rho^j_{p+1} - \rho^j_p| < toll)$
break
endif
end

end,

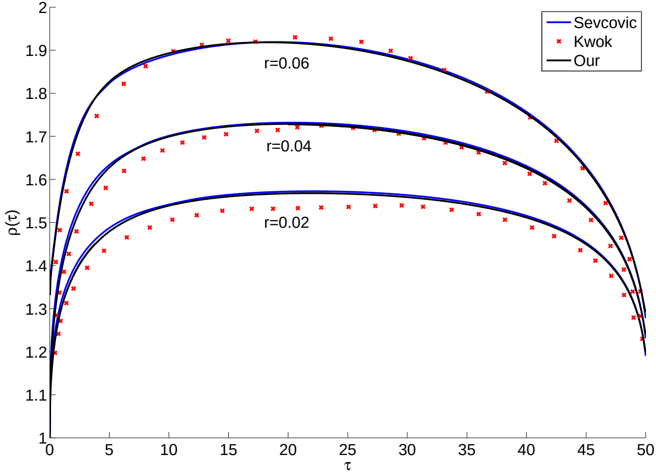
A comparison of the free boundary position for various σ



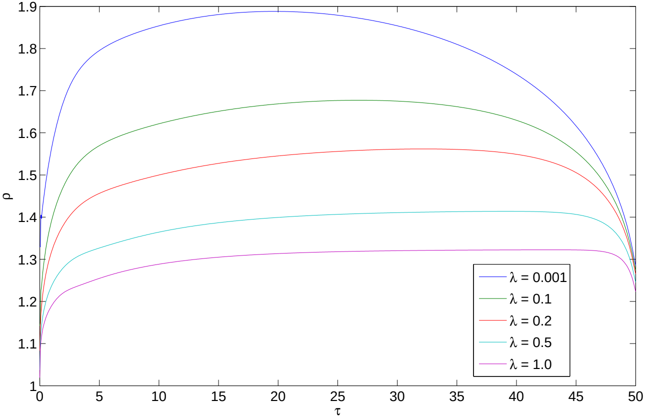
A comparison of the free boundary position for various small σ



A comparison position of the free boundary position for various r



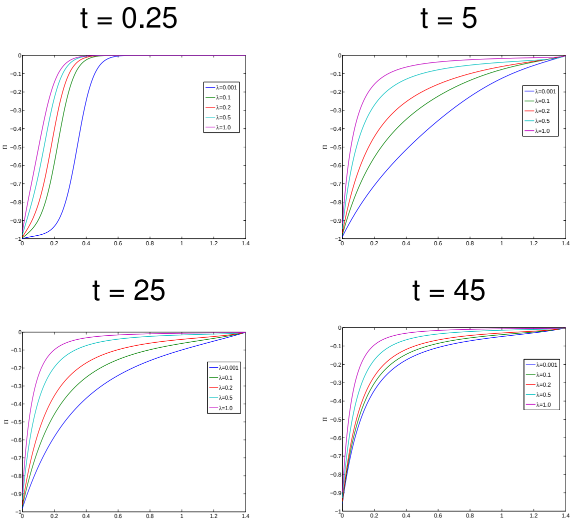
A comparison of the free boundary position for exponentially weighted Asian options



Experimental order of convergence for ρ_λ

λ	$\|\|\rho_\lambda - \rho_\infty\|\|_\infty$	α_∞
0.001	0.888104	–
0.1	0.677320	0.058835
0.2	0.561828	0.269710
0.5	0.413783	0.333796
1.0	0.320136	0.370188
2.0	0.247010	0.374115
3.0	0.212705	0.368769
4.0	0.191862	0.358490
5.0	0.177658	0.344686
10.0	0.147227	0.271064
20.0	0.113350	0.377261
30.0	0.094150	0.457702

Exponetially weighted averaged options



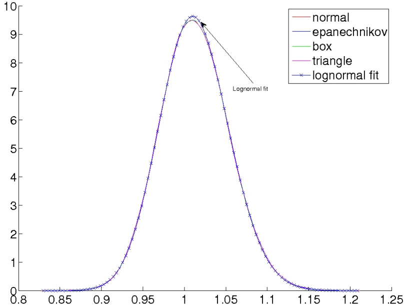
Thank you

Thanks for your attention!

Arithmetic rate Asian options

Payoff diagram

$$V(S, A, T) = (A - X)^{+}$$



Moments of A_T

First moment

$$\mathbb{E}[A_T] = S_0 \frac{\lambda}{\lambda + r} \frac{e^{rT} - e^{-\lambda T}}{1 - e^{-\lambda T}}$$

Second moment

$$\mathbb{E}[A_T^2] = e^{-2\lambda T} \frac{S_0^2}{k^2} \frac{2}{\tilde{\alpha}} \left[\frac{\exp(\tilde{\beta}) - \exp(\tilde{\alpha})}{\tilde{\beta} - \tilde{\alpha}} - \frac{\exp(\tilde{\beta}) - 1}{\tilde{\beta}} \right]$$

$\tilde{\alpha} = (r + \lambda)T$, $\tilde{\beta} = 2(r + \frac{1}{2}\sigma^2 + \lambda)T$, $k = \frac{1}{T} \int_0^T \exp(-\lambda\xi) d\xi$

Approximate formula

$$\begin{aligned} V(S,0) &= e^{-rT}\mathbb{E}_Q[(A_T - X)^+] \\ &= e^{-rT}\int_X^\infty (x - X)\frac{1}{x\chi\sqrt{2\pi}}\exp\left[-\frac{(\ln(x)-\varphi)^2}{2\chi^2}\right]\,dx \end{aligned}$$

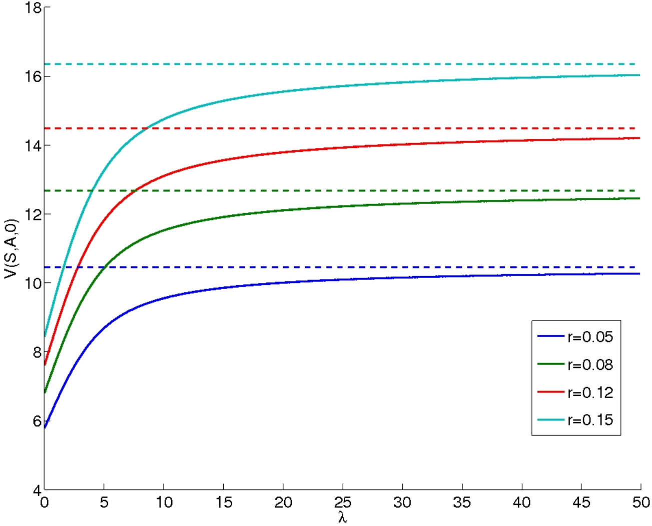
where

$$\begin{aligned} \varphi &= 2\ln(E[A_T]) - \frac{1}{2}\ln E[A_T^2], \\ \chi^2 &= \ln\frac{E[A_T^2]}{(E[A_T])^2}. \end{aligned}$$

σ	X	r	RS-PDE	T-LB	T-UB	AA	LN	MC
0.05	95	0.05	7.157	7.1777	7.1779	7.1794	7.17802	7.14323
0.05	100	0.05	2.621	2.7162	2.7162	2.7279	2.72574	2.70776
0.05	105	0.05	0.439	0.3372	0.3374	0.3257	0.34352	0.33706
0.05	95	0.09	8.823	8.8088	8.8089	8.8091	8.80888	8.87211
0.05	100	0.09	4.185	4.3082	4.3084	4.3173	4.31292	4.31100
0.05	105	0.09	1.011	0.9583	0.9585	0.9561	0.96888	0.94235
0.05	95	0.15	11.090	11.0941	11.0943	11.0941	11.09409	11.13954
0.05	100	0.15	6.777	6.7944	6.7946	6.7963	6.79500	6.78619
0.05	105	0.15	2.639	2.7444	2.7446	2.7559	2.75309	2.73807
0.10	90	0.05	11.942	11.9511	11.9523	11.9666	11.95337	11.82228
0.10	100	0.05	3.624	3.6413	3.6416	3.6725	3.64798	3.64981
0.10	110	0.05	0.359	0.3311	0.3322	0.2855	0.32426	0.34403
0.10	90	0.09	13.382	13.3852	13.3862	13.3935	13.38630	13.43389
0.10	100	0.09	4.887	4.9151	4.9154	4.9597	4.92349	4.93918
0.10	110	0.09	0.659	0.6301	0.6310	0.5840	0.62376	0.65125
0.10	90	0.15	15.398	15.3988	15.3995	15.4015	15.39906	15.48592
0.10	100	0.15	7.000	7.0277	7.0286	7.0707	7.03506	7.03506
0.10	110	0.15	1.430	1.4133	1.4143	1.3901	1.41130	1.35585
0.20	90	0.05	12.589	12.5956	12.6008	12.7837	12.62990	12.51913
0.20	100	0.05	5.760	5.7627	5.7645	5.8330	5.78310	5.78487
0.20	110	0.05	1.996	1.9892	1.9927	1.8322	1.97131	2.00315
0.20	90	0.09	13.825	13.8312	13.8373	14.0072	13.86178	13.83059
0.20	100	0.09	6.773	6.7770	6.7787	6.8915	6.80379	6.83882
0.20	110	0.09	2.551	2.5455	2.5486	2.4269	2.53478	2.57984
0.20	90	0.15	15.636	15.6416	15.6491	15.7898	15.66540	15.69377
0.20	100	0.15	8.402	8.4085	8.4105	8.5691	8.44099	8.40859
0.20	110	0.15	3.558	3.5547	3.5578	3.5098	3.55665	3.55116

A comparison of different methods $S_0 = 100$, $T = 1$. RS–PDE are values obtained by solve PDE (Roger a Shi)

T–LB and T–UB are lower and upper bounds Thompson (2000), AA is analytical approximations Zhang.


18
16
14
12
10
8
6
4
V(S,A,0)
0
5
10
15
20
25
30
35
40
45
50
λ
r=0.05
r=0.08
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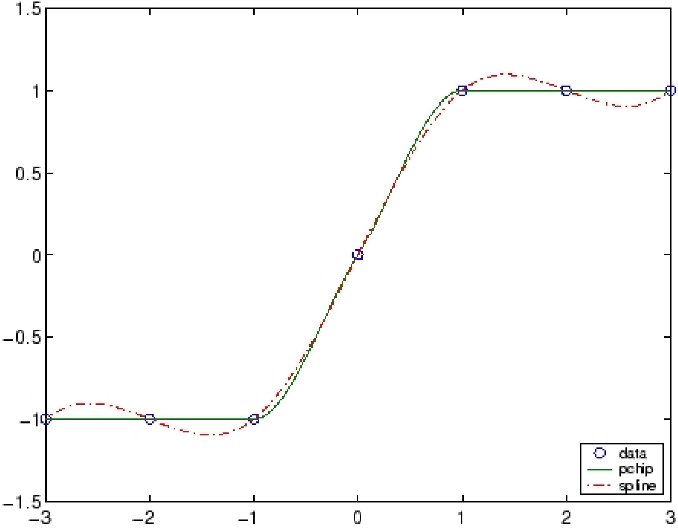
r=0.15


1.5
1
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spline